

EIGENVALUES OF ROTATIONS AND BRAIDS IN SPHERICAL FUSION CATEGORIES

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ABSTRACT. We give formulas for the multiplicities of eigenvalues of generalized rotation operators in terms of generalized Frobenius-Schur indicators in a semi-simple spherical tensor category \mathcal{C} . In particular, this implies for a finite depth planar algebra, the entire collection of rotation eigenvalues can be computed from the fusion rules and the traces of rotation at finitely many depths. If \mathcal{C} is also braided, these formulas yield the multiplicities of eigenvalues for a large class of braids in the associated braid group representations. When \mathcal{C} is modular, this allows one to determine the eigenvalues and multiplicities of braids in terms of just the S and T matrices.

1. INTRODUCTION

Fusion categories are ubiquitous in many areas of mathematics that require a description of symmetry more general than group symmetry. For example, fusion categories essentially classify fully extended 3-dimensional topological field theories [CLDS13], and arise as the representation categories of conformal nets, vertex operator algebras [KLM01], and quantum groups at roots of unity [RT91]. The symmetry of a finite index, finite depth subfactor as described by its standard invariant, can also be understood from a categorical point of view [Jon98], [Müg03a]. The role that fusion categories play as a unifying language in “quantum” mathematics justifies the statement that fusion categories describe *quantum symmetries*.

Fusion categories come in many flavors. In particular they can have a ribbon structure, which enriches the category with braid group symmetry, and allows for the construction of link invariants similar in spirit to the Jones polynomial. A particularly important class of these are the *modular* categories. Given a spherical fusion category \mathcal{C} , its Drinfeld center $Z(\mathcal{C})$ is a canonically associated modular category. $Z(\mathcal{C})$ provides important information about the entire Morita class of \mathcal{C} , and any fusion category can be recovered as the modules of a certain commutative algebra in its center. Thus in some sense, the center contains all the information about \mathcal{C} and *more*, and thus modular categories (Drinfeld centers in particular) provide an important lens through which to view general fusion categories.

Given an object in a pivotal fusion category $a \in \mathcal{C}$ and an object in its center $b \in Z(\mathcal{C})$, Ng and Schauenburg introduced *generalized rotation operators* in [NS10], $\rho_{n,a}^b : \mathcal{C}(b, a^n) \rightarrow \mathcal{C}(b, a^n)$, which are graphically given by a rotation. Their traces, called *generalized Frobenius-Schur (GFS) indicators*, have proved to be important in the study modular categories. They have been used to prove that the corresponding representations of $SL_2(\mathbb{Z})$ factor through the finite groups $SL_2(\mathbb{Z}/M\mathbb{Z})$ where M is the order of the T -matrix, called the *conductor* of \mathcal{C} . GFS indicators are also the main technical ingredient in the proof of rank finiteness for modular categories [BNRW16].

On the other hand, specializing to $b = 1$, we obtain the *higher Frobenius-Schur indicators*, which generalize the Frobenius-Schur indicators of irreducible representations of finite

groups. These were introduced for spherical fusion categories earlier by Ng and Schauenberg [NS07b], generalizing work for Hopf algebras (see [KSZ06] for an overview). The generalized rotation operators in this setting are essentially the same as the rotation operators in Jones' theory of planar algebras [Jon98]. The rotation operator has played an important role in the construction and obstruction theory of planar algebras. In particular understanding rotation eigenvalues in relation to other data, such as the structure of the principal graph, has been crucial to the small index classification program (for an overview, see [Jon01], [JMS]).

While there is a great deal of theory regarding the *traces* of the generalized rotations and how to compute them from basic categorical data, their actual eigenvalues and their multiplicities have been relatively inaccessible, except in special cases. In this note we will give formulas for the multiplicities of the eigenvalues of the generalized rotation operators in terms of the GFS indicators. The basic idea is to use Galois actions on cyclotomic fields, and properties of GFS indicators for semi-simple pivotal categories to derive the traces of the powers of the generalized rotations. Then a finite Fourier transform yields the desired formulas. We have two quite different applications for this. The first is to indicator rigidity for categorifications of fusion rings, where we show that if two categorifications of a fusion ring have the same higher Frobenius-Schur indicator data, then the planar algebras constructed from them are isomorphic to annular Temperley-Lieb-Jones modules [Jon01]. The second application is to the computation of eigenvalues of braids in ribbon fusion categories, where we give a formula for the multiplicities of eigenvalues of a large class of braids in terms of generalized Frobenius-Schur indicators. In particular, when \mathcal{C} is modular, our formulas for the multiplicities can be written in terms of just S and T .

We now elaborate on our results. Let \mathcal{C} be a semi-simple spherical tensor category with fusion tensor N and $a \in \mathcal{C}$. For any irreducible object $b \in Z(\mathcal{C})$ in the Drinfeld center, let $\rho_{n,a}^b : \mathcal{C}(b, a^n) \rightarrow \mathcal{C}(b, a^n)$ be the generalized rotation operator (see Section 3 below). Let $\nu_{n,k}^b(a) := \text{Tr}((\rho_{n,a}^b)^k)$ for $n > 0$, $k \in \mathbb{Z}$ denote the *generalized Frobenius-Schur indicator*. All the eigenvalues of the rotation operator are roots of θ_b^{-1} , so renormalizing by an n^{th} -root of θ_b gives an operator whose eigenvalues are roots of unity. Thus its eigenspaces are isotypic components for the corresponding cyclic group representation. Taking inner products of characters and re-normalizing gives us the following proposition:

Proposition 1.1. *Let \mathcal{C} be a spherical tensor category. For any $a \in \mathcal{C}$, and $b \in \text{Irr}(Z(\mathcal{C}))$, the eigenvalues of $\rho_{n,a}^b$ are n^{th} roots of θ_b^{-1} . For such an ω , its multiplicity as an eigenvalue is $P_{n,a}^b(\omega^{-1})$, where*

$$P_{n,a}^b(x) := \sum_{k=0}^{n-1} \frac{\nu_{n,k}^b(a)}{n} x^k$$

In general, if $b \in Z(\mathcal{C})$ is not irreducible but is semi-simple, we can define

$$K_{n,a}^b(\omega) := \sum_{c \in \text{Irr}(Z(\mathcal{C}))} \delta(\omega^n, \theta_c) \dim Z(\mathcal{C})(c, b) P_{n,a}^c(\omega^{-1}).$$

This gives the eigenvalue multiplicities for $\rho_{n,a}^b$. Thus to compute $P_{n,a}^b$, we need to compute the generalized Frobenius-Schur indicators. Ng and Schauenburg provide a method for doing so in terms of modular data for the center and the forgetful functor matrix when \mathcal{C} is a spherical fusion category.

It is desirable, however, to be able to compute the rotation eigenvalues from as little information as possible, since there are many situations where the full modular data for the

center is not easily accessible. For example, in [GM16], Gannon and Morrison describe a procedure for computing possible modular data for the center of categorifications of a given fusion ring. Their algorithm produces finitely many possibilities for the forgetful functor matrix and the T -matrix effectively, while finding the S -matrix requires solving a generally large system of quadratics which may or may not be possible. But we can compute the higher Frobenius-Schur indicators $\nu_k(a) := \nu_{n,1}^1(a)$ from just the T -matrix and the forgetful functor matrix, so it would be very useful to be able to determine *all* the eigenvalues from this information. There are other similar situations in which we can find partial information about S and T sufficient to compute other GFS indicators $\nu_n^b(a) := \nu_{n,1}^b(a)$, and would like to be able to determine all the rotation eigenvalues from this information.

Motivated by this sort of situation, we show how to compute $\nu_{n,m}^b(a)$ for all m from the information for $m = 1$. The main idea is to renormalize the rotation operators so it's eigenvalues are n^{th} roots of unity, and then use the Galois group actions on the appropriate cyclotomic fields. For $\text{GCD}(k, n) = 1$, let $\alpha_{k,n}$ denote the Frobenius element in the Galois group of the n^{th} cyclotomic extension of \mathbb{Q} , which raises primitive roots of unity to the k^{th} power.

Proposition 1.2. *For $1 \leq k \leq n - 1$*

$$\nu_{n,k}^b(a) = \theta_b^{-\frac{k}{n}} \alpha_{\frac{k}{g}, \frac{n}{g}} \left(\theta_b^{\frac{g}{n}} \nu_{\frac{n}{g}, 1}^b(a^g) \right) \quad \text{where } g = \text{GCD}(k, n)$$

Our first application is to describe an indicator rigidity property. In general, rigidity phenomenon describe situations where you get more for less: some a-priori weak invariants being equal implies some a-priori stronger invariants are equal. Indicator rigidity is this phenomenon applied to indicators, viewed as invariants of spherical categorifications of a given fusion ring. The collection of higher FS indicators $\nu_k(a) := \nu_{k,1}^1(a)$ for all isomorphism classes of simple objects $a \in \mathcal{C}$ and all $k \in \mathbb{N}$, is called the *higher Frobenius-Schur data* of the category \mathcal{C} (HFS data for short). If \mathcal{C} is a fusion category, the sequence is periodic (with period the conductor of the center), so this is a finite set of data. This data has been an important invariant for distinguishing categorifications of a given fusion ring. For example, categorifications of Tambara-Yamagami fusion rings are completely distinguished by their HFS indicator data [BJ15], and in fact, this is true in many examples. Ng has asked whether the HFS data is a complete invariant of categorifications of any fusion ring, a phenomenon which we call *indicator rigidity*. We observe a negative answer to this question: the quadratic Haagerup fusion ring with $G = \mathbb{Z}_3$ admits two inequivalent, unitary (hence spherical) categorifications with the same HFS data.

However, the ability to compute rotation eigenvalues from indicator data can be rephrased as a somewhat surprising rigidity phenomenon. A *planar algebra* is a commutative algebra object in the annular representation category $\text{Rep}(\mathbf{ATLJ})$ of the appropriate Temperley-Lieb-Jones categories. Associated to an object in a category $a \in \mathcal{C}$, one can construct a planar algebra \mathcal{P}_a , whose weight n vector spaces are $\mathcal{C}(1, (a \otimes \bar{a})^n)$. If $a \otimes \bar{a}$ tensor generates, then one can recover \mathcal{C} . Suppose R is a fusion ring, and \mathcal{C} and \mathcal{D} are spherical categorifications of R . Saying that \mathcal{C} and \mathcal{D} categorify R means we have an identification of isomorphism classes of objects \mathcal{C} and \mathcal{D} with positive integral elements of R inducing fusion ring isomorphisms. For $a \in \mathcal{C}$, $b \in \mathcal{D}$, we say $a \sim b$ they are equivalent under this identification.

Proposition 1.3. *(Annular indicator rigidity) The set of rotational eigenvalues of a finite depth planar algebra is determined by the fusion rules of the category, and the HFS data. In*

particular, if the HFS data for two spherical categorifications \mathcal{C} and \mathcal{D} of a given fusion ring are the same, if $a \sim b$ then $\mathcal{P}_a \cong \mathcal{P}_b$ as annular representations of **TLJ**.

Since every fusion category is tensor generated by an object of the form $a \otimes \bar{a}$, and \mathcal{P}_a recovers the category, the failure of full indicator rigidity can be rephrased in terms of non-uniqueness of commutative algebra structures on a given object in $\text{Rep}(\mathbf{ATLJ})$. Also, this gives a possible approach to determining for which class of fusion rings the indicator data *does* distinguish categorifications. For example, it is possible that from general considerations one can determine that certain types objects in $\text{Rep}(\mathbf{ATLJ})$ admit *at most one* commutative algebra structure, which would imply a indicator rigidity for any underlying fusion ring.

Now we discuss our second application, which lies in a different direction. Given an object a in a ribbon fusion category \mathcal{C} , we have a family of representations of the braid groups, $\pi_{n,a} : B_n \rightarrow \text{End}(a^n)$. From the eigenvalues of the generalized rotation operators, we can compute the eigenvalues for a fairly large class of braids in the associated braid group representations. This class of braids includes the braid group generators and the periodic braids. As demonstrated in [TW01, RT10], the eigenvalues of the braid generator may contain a large amount of information about the associated representation in low dimensions. Indeed, small representations of B_3 are essentially determined by the spectrum of the braid group generator [TW01].

To put this result into context, for $a \in \mathcal{C}$, let d be the number of simple objects in $a \otimes a$ and let $\Lambda = \{\lambda_1, \dots, \lambda_d\}$ be the eigenvalues of the braiding in $\text{End}(a^2)$. From the axioms of ribbon fusion categories one can work out the set $\Lambda^2 = \{\lambda_1^2, \dots, \lambda_d^2\}$ and their multiplicities entirely from the T -matrix and fusion rules (hence also from S and T when \mathcal{C} is modular). In many cases related to quantum groups, the actual set of eigenvalues (the spectrum) for the braiding can be worked out using explicit forms of the braid matrix, but until now, there has been no general procedure for finding which square roots occur, and with what multiplicity. But in fact, many interesting modular categories are Drinfeld centers of exotic fusion categories, which don't appear to come from quantum groups in any obvious way (see [HRW08]), and we anticipate the utility of our formulas in these cases.

To be more precise, suppose that \mathcal{C} is a ribbon category and fix simple objects $a, b \in \mathcal{C}$. Define the element $A_{l,m}^n \in \text{End}(a^n)$ by

$$A_{l,m}^n :=$$

Identify $\mathcal{C} \boxtimes \tilde{\mathcal{C}}$ with its image under the canonical functor $G : \mathcal{C} \boxtimes \tilde{\mathcal{C}} \rightarrow Z(\mathcal{C})$. The braid group acts on the multiplicity space $\mathcal{C}(b, a^n)$ and we call the corresponding representation $\pi_{n,a}^b$.

Corollary 1.4. *The operators $\pi_{n,a}^b(A_{l,m}^n)$ are diagonalizable, and their eigenvalues are of the form $\theta_a^{-1}\omega$ for ω and $(n - (m + l))^{th}$ root of θ_b . The eigenvalue $\theta_a^{-1}\omega$ occurs with multiplicity $K_{n-(m+l),a}^{\bar{a}^{l+m} \boxtimes \bar{b}}(\omega)$.*

The element $A_{0,0}^2$ is the standard braid generator. Therefore, if we combine theorem 1.1 with corollary 1.4, then we have

Corollary 1.5. *The eigenvalues of the braid generator acting on $\mathcal{C}(b, a^2)$ are square roots of $\frac{\theta_b}{\theta_a^2}$, and for ω a square root θ_b , $\theta_a^{-1}\omega$ occurs with multiplicity*

$$P_{2,a}^{1 \boxtimes \bar{b}}(\omega^{-1}) = \frac{1}{2} \left(\omega^{-1} \nu_{2,1}^{1 \boxtimes \bar{b}}(a) + N_{a,a}^b \right)$$

When \mathcal{C} is modular, we can express the generalized Frobenius-Schur indicators in terms of the modular data. Therefore, corollary 1.5 specializes to

Corollary 1.6. *If \mathcal{C} is modular with modular data S, T then*

$$P_{2,a}^{1 \boxtimes \bar{b}}(x) = \frac{1}{2} \left(x \sum_{d,e \in \text{Irr}(\mathcal{C})} \left(\frac{\theta_d}{\theta_e} \right)^2 S_{1,d} S_{\bar{b},e} N_{d,e}^a + N_{a,a}^b \right)$$

Remark 1.7. All of our results actually hold when \mathcal{C} is a semi-simple spherical tensor category which may have infinite isomorphism classes of objects. Also, in this case, one can replace “simple object in the center” with “locally finite, irreducible object” in $Z(\text{Ind-}\mathcal{C})$, where a locally finite object $A \in \text{Ind-}\mathcal{C}$ satisfies $\dim(\text{Ind-}\mathcal{C}(A, a)) < \infty$ for all $a \in \mathcal{C}$. We primarily restrict our attention to fusion categories since we have greater access to the higher indicators, but we have been careful to prove our results in a way that generalize easily to the non-fusion case.

2. PRELIMINARIES

In this paper, we only consider semi-simple, rigid tensor categories. Of particular interest will be *fusion categories*:

Definition 2.1. A *fusion category* is a semi-simple rigid tensor category with a finite number of simple objects and a simple unit.

We assume, for convenience, that our fusion categories are strict, which can be done without loss of generality by MacLane’s strictness theorem as explained in chapter 2 of [EGNO15]. If a, b, c are simple objects in our category, the *fusion rule* is the collection of numbers $N_{a,b}^c := \dim \mathcal{C}(a \otimes b, c)$. These numbers depend on the objects only up to equivalence. In general, we use the notation $N_{a_1, \dots, a_n}^b := \dim \mathcal{C}(a_1 \otimes \dots \otimes a_n, b)$. These larger multiplicities can be computed from the fusion rule in the following way:

$$N_{a,b,c}^d = \sum_{e \in \text{Irr}(\mathcal{C})} N_{a,e}^d N_{b,e}^c.$$

We also use the notation $a^n := a \otimes \dots \otimes a$ so that, for example, $N_{a,a,a}^d = N_{a^3}^d$.

Definition 2.2. A *pivotal structure* on a rigid tensor category is a choice of natural monoidal isomorphism from the double dual functor to the identity.

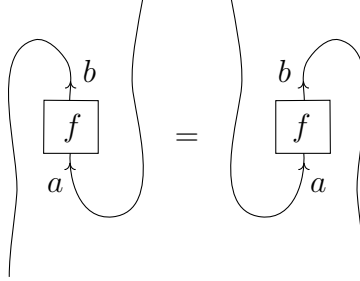
With a pivotal structure one can define left and right traces. We can always choose our pivotal structure to be *strictly pivotal* in the sense that we can pick our duality functor so that the natural isomorphism from the double dual to the identity functor is the identity [NS07a]. This means that for each object $a \in \mathcal{C}$, we have an object \bar{a} and morphisms $\text{coev}_a \in \mathcal{C}(1, \bar{a} \otimes a)$, $\text{coev}_{\bar{a}} \in \mathcal{C}(1, a \otimes \bar{a})$, $\text{ev}_a \in \mathcal{C}(\bar{a} \otimes a, 1)$, $\text{ev}_{\bar{a}} \in \mathcal{C}(a \otimes \bar{a}, 1)$, given graphically by oriented cups and caps, where as usual, these cups and caps satisfy the so-called zig-zag, or duality, equations (see, for example, [Müg03a] for duality and graphical calculus). For $f \in \mathcal{C}(a, b)$, define left rotation by π as

$$\bar{f}^l := (\text{ev}_b \otimes 1_{\bar{a}}) \circ (1_{\bar{b}} \otimes f \otimes 1_{\bar{a}}) \circ (1_{\bar{b}} \otimes \text{coev}_{\bar{a}}) \in \mathcal{C}(\bar{b}, \bar{a})$$

and right rotation by π as

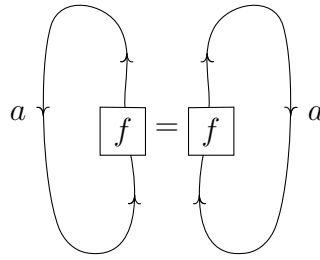
$$\bar{f}^r := (1_{\bar{a}} \otimes \text{ev}_{\bar{b}}) \circ (1_{\bar{a}} \otimes f \otimes 1_{\bar{b}}) \circ (\text{coev}_a \otimes 1_{\bar{b}})$$

Strictly pivotal means $\bar{f}^l = \bar{f}^r$, so that we can unambiguously define $\bar{f} = \bar{f}^l = \bar{f}^r$. Then it is easy to check that the duality functor \bar{f} satisfies $\bar{\bar{f}} = f$. Strictly pivotal categories allow us to make full use of the graphical calculus. **Warning:** Our graphical calculus convention is that diagrams are read bottom to top, the opposite convention to [NS07b, NS10], but the same as most other sources such as [Müg03a, Müg03b]. Strictly pivotal can be translated graphically as



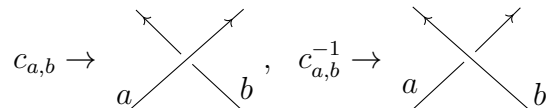
Definition 2.3. A pivotal structure is called *spherical* if the linear functionals $\text{Tr}_l(f) = \text{ev}_a \circ (1_{\bar{a}} \otimes f) \circ \text{coev}_a$ and $\text{Tr}_r = \text{ev}_{\bar{a}} \circ (f \otimes 1_{\bar{a}}) \circ \text{coev}_{\bar{a}}$ are equal for all objects a and $f \in \mathcal{C}(a, a)$.

Graphically this condition is pictured as




Definition 2.4. A *braided tensor category* is a tensor category which has, for each pair of objects, an isomorphism $c_{a,b} \in \mathcal{C}(a \otimes b, b \otimes a)$ satisfies the hexagon relations. These isomorphisms must be natural in a and b .

Graphically, the braiding is represented by



Definition 2.5. If a braided tensor category \mathcal{C} is also spherical, this equips \mathcal{C} with the structure of a *ribbon category*, and every ribbon fusion category is of this form.

The ribbon structure manifests in isomorphisms $\theta_a \in \mathcal{C}(a, a)$ called *twists* defined, from a spherical braided structure, by

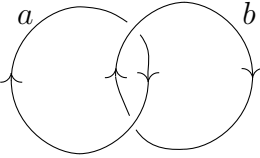
$$\theta_a := (1_a \otimes \text{ev}_a) \circ c_{a,a} \circ (1_a \otimes \text{coev}_a) =$$


In particular, if a is simple, θ_a is a scalar times 1_a (we often abuse notation and write θ_a for this scalar). These twist isomorphisms satisfy the following ribbon axiom:

$$\theta_{a \otimes b} = c_{b,a} \circ c_{a,b} \circ (\theta_a \otimes \theta_b)$$

Let n be the number of simple objects in \mathcal{C} . We define the $n \times n$ matrices

$$\hat{T}_{a,b} := \delta_{a,b} \theta_a$$

$$\hat{S}_{a,b} := \text{tr} (c_{b,\bar{a}} \circ c_{\bar{a},b}) =$$


Let d_a be the dimension of the simple object a . We define the global dimension

$$D = \sum_{a \in \text{Irr}(\mathcal{C})} d_a^2$$

and set

$$T = \hat{T} \quad S = \frac{1}{D} \hat{S}.$$

Definition 2.6. A ribbon fusion category is called *modular* if the matrix S is invertible.

We define the *charge conjugation matrix* $C = \delta(\bar{a}, b)$, and set $\xi = \sqrt{\frac{\sum_a \theta_a d_a^2}{\sum_a \theta_a^{-1} d_a^2}}$ (which is always defined if \mathcal{C} is modular). Then the following relations are satisfied:

$$(ST)^3 = \xi S^2, \quad S^2 = C, \quad C^2 = 1, \quad CS = SC, \quad CT = TC.$$

These equations imply that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow S \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow T$$

is a projective representations of $SL_2(\mathbb{Z})$. Vafa's theorem asserts that all twists are roots of unity, hence there exists an M such that $T^M = 1$, which we call the *conductor* of the category. We refer the reader to [BK01], chapter 3 for all of the above results. It was proved that the representation of $SL_2(\mathbb{Z})$ factors through $SL_2(\mathbb{Z}_M)$ [NS10]. Furthermore, the entries of S, T lie in the cyclotomic field $\mathbb{Q}(e^{2\pi i/M})$, and there are many number theoretic constraints on the modular data. A lot of information about \mathcal{C} is contained in S and T . For example,

the *Verlinde formula* allows one to express the fusion rules of the category in terms of the S matrix:

$$N_{c,d}^a = \sum_{e \in \text{Irr}(\mathcal{C})} \frac{S_{c,e} S_{d,e} \overline{S}_{a,e}}{S_{1,e}}$$

Given an arbitrary spherical (semi-simple) tensor category, one can construct the Drinfeld center $Z(\mathcal{C})$ as \mathcal{C} - \mathcal{C} bi-modular endofunctors on \mathcal{C} (viewed as a \mathcal{C} - \mathcal{C} bi-module category). Given such an endo functor F , $F(1)$ is an object in \mathcal{C} , and the bimodule morphisms yield natural isomorphisms $e_{F,a} : F(1) \otimes a \rightarrow a \otimes F(1)$ for all objects a satisfying various compatibility conditions, which we call *half-braidings*. The Drinfeld center is completely determined by these half braidings as explained in see [EGNO15], Definition 7.13.1. The Drinfeld center is always modular and categorifies the center of an algebra. We have the forgetful functor $\text{Forget} : Z(\mathcal{C}) \rightarrow \mathcal{C}$, given in the endo-functor picture by sending F to $F(1)$. This functor has a left adjoint called the *central induction functor*. If \mathcal{C} is braided, then we can define $\tilde{\mathcal{C}} = \mathcal{C}$ as tensor categories but with a new braiding

$$\tilde{c}_{a,b} := c_{b,a}^{-1}.$$

This new braided category is also ribbon fusion, with S matrix $\tilde{S}_{a,b} = S_{a,b}^{-1} = S_{\bar{a},b}$, and T -matrix $\theta_{\bar{a}} = \theta_a^{-1}$. For a ribbon fusion category, we always have a braided tensor functor $G : \mathcal{C} \boxtimes \tilde{\mathcal{C}} \rightarrow Z(\mathcal{C})$, where here \boxtimes is the Deligne tensor product. Theorem 7.10 in [Müg03b] says that if \mathcal{C} is modular, then G is a braided equivalence of tensor categories. Using G , we can identify $Z(\mathcal{C})$ with $\mathcal{C} \boxtimes \tilde{\mathcal{C}}$, and under this identification, the functor $\text{Forget} : Z(\mathcal{C}) \rightarrow \mathcal{C}$ is equivalent to the functor which sends $X \boxtimes Y \rightarrow X \otimes Y \in \mathcal{C}$, and simply forgets about the braiding. Thus in the modular case, we can recover S and T for $Z(\mathcal{C})$ from S and T of \mathcal{C} , as well as the structure of the forgetful functor $\text{Forget} : Z(\mathcal{C}) \rightarrow \mathcal{C}$. In particular, since the simple objects of $Z(\mathcal{C})$ are indexed by equivalence classes of $a \boxtimes \tilde{b}$, We have

$$(1) \quad S_{a \boxtimes \tilde{b}, c \boxtimes \tilde{d}} = S_{a,c} \tilde{S}_{b,d} = S_{a,c} S_{\bar{b},d}$$

$$\theta_{a \boxtimes \tilde{b}} = \frac{\theta_a}{\theta_b}$$

Notice that $G|_{\mathcal{C} \boxtimes 1}$ is a tensor equivalence onto its image, hence any ribbon fusion category embeds as a full subcategory into a modular one. In particular, this implies the T -matrix of a ribbon fusion category also has finite order, which we also call the conductor, or simply the order of the T -matrix.

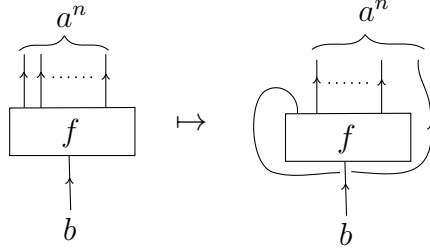
The braid group acts on the multiplicity space $\mathcal{C}(b, a^n)$ and we call the representation $\pi_{a,n}^b$. Given an object $a \in \mathcal{C}$ in a ribbon tensor category, we have a homomorphism $\pi_{a,n} : B_n \rightarrow \text{End}(a^n)$. The tensor structure on \mathcal{C} gives us an inclusion $\text{End}(a^n) \rightarrow \text{End}(a^{n+1})$ which is compatible with the natural inclusion $B_n \rightarrow B_{n+1}$. Taking colimits gives a homomorphism $\pi_a : B_\infty \rightarrow \cup_n \text{End}(a^n)$. This implies that the Spectrum of $\pi_{a,n}(g)$ only depends on the conjugacy class of g inside B_∞ . In other words, the representations of our braid group are “local”. Furthermore, suppose a braid x is conjugate in B_∞ to some braid $y \in B_n$. Then to determine the spectrum of $\pi_a(x)$, it suffices to determine the spectrum of $\pi_{a,n}^b(y)$ acting on $\mathcal{C}(b, a^n)$ for all $b \in \text{Irr}(\mathcal{C})$. In other words $\text{Spec}(\pi_a(x)) = \bigcup_{b \in \text{Irr}(\mathcal{C})} \text{Spec}(\pi_{a,n}^b(y))$.

3. GENERALIZED ROTATION

For a spherical tensor category \mathcal{C} , we recall that an object in the Drinfeld center $Z(\mathcal{C})$ is determined by an object $b \in \mathcal{C}$ and a half braiding. For $a \in \mathcal{C}$ and $m \in \mathbb{N}$, we define the *rotation operator* $\rho_{n,a}^b : \mathcal{C}(b, a^{\otimes n}) \rightarrow \mathcal{C}(b, a^{\otimes n})$ by

$$\rho_{n,a}^b(f) := (\text{ev}_a \otimes 1_{a^{\otimes n}}) \circ (1_{\bar{a}} \otimes f \otimes 1_a) \circ (e_{b,\bar{a}} \otimes 1_a) \circ (1_b \otimes \text{coev}_a)$$

In pictures, this is given by



We notice several properties of the rotation operators:

- (1) $(\rho_{n,a}^b)^n = \theta_b^{-1} \text{id}$.
- (2) $\rho_{n,a}^b$ is diagonalizable

Ng and Schauenburg define the *generalized Frobenius-Schur indicators* for $n > 0$ and $l \in \mathbb{Z}$:

$$\nu_{n,k}^b(a) := \text{Tr}((\rho_{n,a}^b)^k).$$

The trace is taken as a linear map. These numbers have a natural definition for $n \in \mathbb{Z}$ as well, but we won't need them here. [NS10], Ng and Schauenburg use them to show that the representation of $SL_2(\mathbb{Z})$ given by the S and T matrices factors through a representation of $SL_2(\mathbb{Z}_M)$, where M is the order of the T matrix. They are also used in the proof of modular rank finiteness in [BNRW16]. In the next subsection, we will describe how to compute this numbers from the central modular data. For now, here are some useful properties of GFS indicators which are proved in [NS10]:

- (1) $\nu_{m,l}^b(a \oplus d) = \nu_{m,l}^b(a) + \nu_{m,l}^b(d)$ if $\text{GCD}(m, l) = 1$
- (2) $\nu_{m,l}^{b \oplus c}(a) = \nu_{m,l}^b(a) + \nu_{m,l}^c(a)$
- (3) $\nu_{m,l}^b(a) = \nu_{\frac{m}{g}, \frac{l}{g}}^b(a^g)$, where $g = \text{GCD}(m, l)$
- (4) If b is irreducible in $Z(\mathcal{C})$, then $\nu_{m, l+km}^b(a) = \theta_b^{-k} \nu_{m,l}^b(a)$

Suppose we have a diagonalizable operator D on a finite dimensional vector space, whose eigenvalues are N^{th} roots of unity for some N . We can use the Discrete Fourier Transform to recover the collection of eigenvalues knowing the values $\text{Tr}(D^k)$ for $1 \leq k \leq N$. Let ζ be a primitive N^{th} root of unity. For a vector $x := (x_1, \dots, x_N) \in \mathbb{C}^N$, the Discrete Fourier Transform (DFT), written $F(x) = (F(x)_1, \dots, F(x)_N) \in \mathbb{C}^N$, is defined by

$$F(x)_k = \sum_{m=1}^N x_m \zeta^{mk}.$$

The operator F is invertible, and

$$F^{-1}(X)_k := \frac{1}{N} \sum_{m=1}^N X_m \zeta^{-mk}.$$

Let K_m be the dimension of the eigenspace of D corresponding to the root of unity ζ^m . Consider the vector $x = (K_1, \dots, K_M)$. Then since $\text{Tr}(D^k) = \sum_{m=1}^N K_m \zeta^{mk}$, it follows that

$$F(x) = (\text{Tr}(D), \text{Tr}(D^2), \dots, \text{Tr}(D^N)).$$

This implies the m^{th} component of $F^{-1}(\text{Tr}(D), \text{Tr}(D^2), \dots, \text{Tr}(D^N))$ will be the multiplicity K_m with which the eigenvalue ζ^m occurs for the operator D . In short, the inverse DFT, applied the trace of powers vector, yields the eigenvalue multiplicity vector.

For any objects $a \in \mathcal{C}$ and $b \in Z(\mathcal{C})$ irreducible, the eigenvalues of $\rho_{n,a}^b$ are n^{th} roots of θ_b . Define the renormalized rotation operator $\kappa_{n,a}^b = \theta_b^{\frac{1}{n}} \rho_{n,a}^b$, where $\theta_b^{\frac{1}{n}}$ is an arbitrary (but fixed) root of θ_b . Since $(\kappa_{n,a}^b)^n = 1$, its eigenvalues are n^{th} roots of unity, hence we can use the discrete fourier transform to compute eigenvalue multiplicities of the renormalized rotation. Note that $\text{Tr}((\kappa_{n,a}^b)^k) = \theta_b^{\frac{k}{n}} \nu_{n,k}^b(a)$. Thus if ω is a n^{th} root of unity, the multiplicity of ω as an eigenvalue of rotation $\kappa_{n,a}^b$ is given by

$$\frac{1}{n} \left(\sum_{k=1}^n \theta_b^{\frac{k}{n}} \nu_{n,k}^b(a) \omega^{-k} \right) = \frac{1}{n} \left(\sum_{k=1}^n \nu_{n,k}^b(a) (\theta_b^{\frac{1}{n}} \omega^{-1})^k \right)$$

But if the root ω occurs in κ with the same multiplicity as $\theta_b^{-\frac{1}{n}} \omega$ appears in ρ . Putting this all together, we obtain Proposition 1.1.

3.1. Computing GFS indicators: more for less. In [NS10], Ng-Schauenburg explain how to compute the generalized Frobenius Schur indicators from the S and T matrices of the center $Z(\mathcal{C})$ and the $|\text{Irr}(Z(\mathcal{C}))| \times |\text{Irr}(\mathcal{C})|$ matrix A , defined entry-wise by

$$A_{b,a} = \dim(\mathcal{C}(\text{Forget}(b), a)).$$

Let a π denote the $\text{SL}_2(\mathbb{Z})$ representation generated by the S and T matrices, and let $(a, b) \cdot g$ denote the standard (right) action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^{\oplus 2}$. For $m > 0$ and $l \in \mathbb{Z}$ with $\text{GCD}(m, l) = 1$, let $\mathcal{V}_{m,l}$ be the $|\text{Irr}(Z(\mathcal{C}))| \times |\text{Irr}(\mathcal{C})|$ -matrix defined by $(\mathcal{V}_{m,l})_{b,a} = \nu_{m,l}^b(a)$. We have the following formula, due to Ng and Schauenburg, which we've rephrased to suit our purposes:

Proposition 3.1. ([NS10], Theorem 5.4) *Let $\text{GCD}(m, l) = 1$, and $g \in \text{SL}_2(\mathbb{Z})$ such that $(1, 0) \cdot g^{-1} = (m, l)$. Then $\mathcal{V}_{m,l} = \pi(g)A$.*

This shows that if the modular data is at hand, all GFS indicators can be computed, and even that there are only finitely many numbers appearing since π factors through a finite group. The purpose of this section is to show that we can actually compute *all* the GFS indicators from only the “first” higher indicators $\nu_{n,1}^b(a)$. Suppose $n \in \mathbb{N}$ and let ζ_n be a primitive n^{th} root of unity. If $\text{GCD}(k, n) = 1$, let $\alpha_{k,n}$ denote the Galois automorphism on the cyclotomic field $\mathbb{Q}(\zeta_n)$, which acts by $\zeta_n \rightarrow \zeta_n^k$. Let $a \in \mathcal{C}$, and $b \in \text{Irr}(Z(\mathcal{C}))$ be fixed, and pick an n^{th} root of θ_b .

Proposition 3.2. *For $1 \leq k \leq n-1$*

$$\nu_{n,k}^b(a) = \theta_b^{-\frac{k}{n}} \alpha_{\frac{k}{g}, \frac{n}{g}} \left(\theta_b^{\frac{g}{n}} \nu_{\frac{n}{g}, 1}^b(a^g) \right) \quad \text{where } g = \text{GCD}(k, n)$$

Proof. As in the previous section, define the operator $\kappa_{n,a}^b = \theta_b^{\frac{1}{n}} \rho_{n,a}^b$. Then $\kappa_{n,a}^b$ is diagonalizable, and $(\kappa_{n,a}^b)^n = \text{id}$, hence the eigenvalues of κ are powers of ζ_n , and $\text{Tr}(\kappa^l) \in \mathbb{Z}[\zeta]$. Its clear that if $\text{GCD}(k, n) = 1$, then $\theta_b^{\frac{k}{n}} \nu_{n,k}^b(a) = \text{Tr}(\kappa^k) = \alpha_{k,n}(\text{Tr}(\kappa)) = \alpha_{k,n}(\theta_b^{\frac{1}{n}} \nu_{n,1}^b(a))$, hence the

formula is true in this case. It is then easy to see the general result since $\nu_{n,k}^b(a) = \nu_{\frac{n}{g}, \frac{k}{g}}^b(a^g)$ for $g = \text{GCD}(n, k)$. \square

Remark 3.3. Given the underlying \mathcal{C} , if one can compute the induction to the center matrix, as well as T for the center, one can find the higher Frobenius-Schur indicators (namely, $\nu_{m,1}^1(a)$), and then apply our formula to find all the eigenvalues of rotation for all n . Interestingly, given a fusion ring, Gannon and Morrison have outlined an effective procedure for finding a finite set of possibilities for the central induction and T -matrix pairs from the fusion ring alone, which often (but not always) yields a unique such pair [GM16]. Given an induction and T -matrix pair that does not actually correspond to a categorification, there seems to be no a-priori reason why these numbers should live in the right number field suitable for applying the Galois automorphisms in the first place, or that the multiplicities of the eigenvalues should add up to the number as determined by the fusion rules. Checking this could add an additional layer to the procedure of Gannon and Morrison, which could help to rule out extraneous possibilities of S and T pairs, or even to show that certain fusion rings admit no categorifications.

3.2. Indicator rigidity. Let R be a fusion ring. Ocneanu rigidity says that there are only finitely many categorifications of R . In general, one would like to classify all categorifications. While actually constructing categorifications is immensely difficult, an invariant that completely distinguished categorifications (assuming they existed in the first place) would be very useful. A fusion ring R exhibits *indicator rigidity* if the higher FS-indicator data is precisely such an invariant, or in other words, if two spherical categorifications of R have the same HFS data, then they are tensor equivalent. To be more precise, let R be the fusion ring over \mathbb{Z} with canonical basis $R_B = \{b_i\}_{i=1}^n$, and let \mathcal{C} be a spherical categorification of R , meaning we have an identification of $\text{Irr}(\mathcal{C})$ with R_B that induces a fusion ring homomorphism. Let M be the order of the T matrix for $Z(\mathcal{C})$. In this section, we change our notation for simplicity, and denote $\nu_j(a) = \nu_{j,1}^1$, and call these numbers the higher Frobenius-Schur indicators. Then the indicator data for \mathcal{C} is the $M \times n$ matrix $\mathcal{V}_{\mathcal{C}} = (\nu_i(b_j))_{i,j}$. Recall that this captures all the information about the higher indicators, since $\nu_j = \nu_{j+M}^1$ for $j > 0$.

Definition 3.4. A fusion ring R exhibits *indicator rigidity* if for any spherical categorifications \mathcal{C} and \mathcal{D} of R , if $\mathcal{V}_{\mathcal{C}} = \mathcal{V}_{\mathcal{D}}$, then $\mathcal{C} \cong_{\otimes} \mathcal{D}$.

There are many classes of fusion rings which exhibit indicator rigidity. In particular, many group type fusion rings, as well as Tambara-Yamagami fusion rings [BJ15]. Ng has asked whether all fusion rings generated by a simple object exhibit indicator rigidity, but we present a negative answer here from quadratic categories.

We recall the quadratic Haagerup fusion ring H with canonical basis $\{1, g, g^2, \rho, \rho g, \rho g^2\}$, generated by the relations $g^3 = 1$, $g\rho = \rho g^2$ and $\rho^2 = 1 + \rho + \rho g + \rho g^2$.

Proposition 3.5. *The fusion ring H does not exhibit indicator rigidity.*

Proof. Izumi constructed two inequivalent unitary categorifications of this ring. Evans and Gannon parameterize the modular data and induction matrices of this type of category by a group (in this case \mathbb{Z}_{13}) and a quadratic form on this group [EG15]. In, [Tuc15], the third author demonstrates that the indicator data is entirely determined by this form, and from [EG15], Theorem 3, we see that the two unitary categorifications correspond to the same quadratic form. In fact not only are the indicators the same, but the generalized indicators

are the same, since the entirety of the modular data for the center (as well as the induction matrix) is the same. \square

As we've seen, even the much stronger condition of asking all *generalized*-FS indicator data to be the same is not sufficient for general rigidity purposes, by our last counter examples. However, while indicator rigidity is too much to ask for in general, it would be interesting to determine criteria on a fusion ring which would imply indicator rigidity. In this direction, the fact that we can recover the eigenvalues of rotation from the (first) indicators shows that in any planar algebra constructed from \mathcal{C} , the structure of this planar algebra as an annular TLJ representation is completely determined by the HFS data, at least in the unitary case, which we consider from here out.

Recall the strict tensor category $\mathbf{TLJ}(\delta)$, where objects are natural numbers, and morphisms from n to m are given by linear spans of planar non-crossing partitions $n + m$ on points (these is the TLJ diagrams with $2n$ and $2m$ points on bottom/top of the rectangle respectively, with non-crossing pairings between the points). Closed circles evaluate to the number δ . Tensoring is given by addition of natural numbers and horizontal juxtaposition of diagrams. $\mathbf{TLJ}(\delta)$ has a semisimple idempotent completion unless $\delta \in \{2 \cos \frac{\pi}{n} : n \geq 3\}$. From here on we assume δ is semi-simple is a value which makes \mathbf{TLJ} semi-simple, and we will also assume it is understood from context, so we write \mathbf{TLJ} for short.

Another category with richer intrinsic structure is the (affine) annular category of Temperley-Lieb-Jones. Again, objects are given by natural numbers, $\mathbf{ATLJ}(n, m)$ is the set of *annular* non-crossing partitions with n -points on the inner boundary and m points on the outer boundary (for a formal definition, see [JR06] and [Jon01]). This category is quite far away from being semi-simple, and it is not a tensor category. Its representation category, while still being far away from semi-simple, *is* a tensor category, and is even braided (see [DGG14]) In fact, $\text{Rep}(\mathbf{ATLJ})$ is equivalent as a braided tensor category to $Z(\text{Ind-}\mathbf{TLJ}^{\text{op}})$, and the forgetful functor $F : Z(\text{Ind-}\mathbf{TLJ}^{\text{op}}) \rightarrow \text{Ind-}\mathbf{TLJ}^{\text{op}}$ is precisely the restriction of an annular category rep to a category rep.

A representation V of the annular category is called *locally finite* if each V_n is finite dimensional. While the category of all annular representations is huge and wild, we do have a classification of *unitary* locally finite, irreducible representations thanks to [Jon01, JR06]

Proposition 3.6. [Jon01, JR06] *An irreducible, V unitary representation of \mathbf{ATLJ} is classified by*

- (1) *A number $n \geq 0$ called the lowest weight of V such that $V_k = 0$ for $k < n$ and $V_n \neq 0$.*
- (2) *For $n > 0$ unitary scalar $\omega \in S^1$, which is the rotation eigenvalue, and for $n = 0$, a number $0 \leq \mu \leq \delta^2$ which is the eigenvalues of the double closed loop.*

Furthermore, a locally finite, unitary representation of \mathbf{ATLJ} is semi-simple [Jon01], i.e. every such representation is isomorphic to a (possibly infinite) direct sum of irreducible representations (note that at each weight space, only finitely many such irreducible modules can appear by local finiteness).

In fact, every representations can be decomposed $V \cong \bigoplus_n V_n$, where each V_n is a lowest weight representation of \mathbf{ATLJ} . This means that $\mathbf{ATLJ}(n, m)V_n = 0$ for $m < n$. By the above corollary, each V_n for $n > 0$ is classified up to isomorphism by the set of eigenvalues of the rotation operator. The only exception is the weight 0 part, where there is no rotation. This gives us the following corollary:

Corollary 3.7. *Locally finite, unitary representations $V \in \text{Rep}(\mathbf{ATLJ})$ with a fixed weight 0 component V_0 , are classified up to isomorphism by the set of eigenvalues Ω_n of the rotation operator on each V_n , $n > 0$.*

Now, let \mathcal{C} be a spherical tensor category, and let $a \in \mathcal{C}$ (assume for simplicity $d_a \geq 2$). Define the spaces $\mathcal{P}_n^a := \mathcal{C}(1, (a \otimes \bar{a})^n)$. Then the spherical structure endows the collection of spaces $\mathcal{P}_n^a := \mathcal{C}(1, (a \otimes \bar{a})^n)$ with the structure of an object in $\text{Rep}(\mathbf{ATLJ})$ (here $\delta = d_a$). In fact, it is easy to see that \mathcal{P}^a is a *commutative algebra* object in $\text{Rep}(\mathbf{ATLJ})$, where the multiplication maps $\mathcal{P}^a \otimes \mathcal{P}^a \rightarrow \mathcal{P}^a$ are given by evaluation in \mathcal{C} .

Now, suppose R is a fusion ring and \mathcal{C} and \mathcal{D} are categorifications of R . Fix specified isomorphisms of \mathcal{C} and \mathcal{D} to R . We say $a \sim b$ if $[a] = [b] \in R$ for $a \in \mathcal{C}$ and $b \in \mathcal{D}$. Note that in particular, $a \sim b$ implies they have the same dimension.

Proposition 3.8. *Suppose R is a fusion ring and \mathcal{C} and \mathcal{D} are two spherical categorifications of R with $\mathcal{V}_{\mathcal{C}} = \mathcal{V}_{\mathcal{D}}$. If $a \in \mathcal{C}$ and $b \in \mathcal{D}$ with $a \sim b$, and $d_a \geq 2$, then \mathcal{P}_a is isomorphic to \mathcal{P}_b as objects in $\text{Rep}(\mathbf{ATLJ})$.*

Proof. It's clear that the weight 0 components of \mathcal{P}^a and \mathcal{P}^b are both given by the number $d_a^2 = d_b^2$, hence have isomorphic weight 0 components. By Corollary 3.7, it suffices to show that the eigenvalues of the rotation are the same as sets for all non-zero weights. But but since the (first) indicator data is the same, combining by 1.1 and 3.2 yields the result. \square

Note that while our argument was explic

4. COMPUTING EIGENVALUES IN THE BRAID REPRESENTATIONS IN A RIBBON FUSION CATEGORY

Consider for $l, m \geq 0$, $l + m < n$ the elements of the braid group B_n given by

$$A_{l,m}^n :=$$

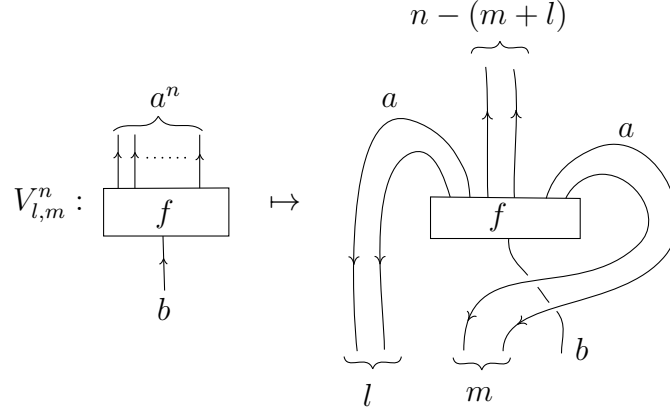
Note that we could define $A_{l,m}^{n,-}$ similarly with over and under crossings switched, and the rest of this section follows from similar arguments. Letting σ_i be the standard braid group generators, inside B_∞ , σ_i is conjugate to $A_{0,0}^2$ and $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ is conjugate to $A_{1,0}^3$.

We can compute the eigenvalues $A_{l,m}^n$ in terms of generalized Frobenius-Schur indicators using the results from the last section. Let \mathcal{C} be ribbon fusion, and let $a, b \in \mathcal{C}$. We define the linear isomorphisms

$$V_{l,m}^n : \mathcal{C}(b, a^{\otimes n}) \rightarrow \mathcal{C}(\bar{a}^{l+m} \otimes b, a^{n-(l+m)})$$

by the following formulas:

$$V_{l,m}^n(f) := (\text{ev}_{\bar{a}^l} \otimes 1_{a^{n-(m+l)}} \otimes \text{ev}_{\bar{a}^m}) \circ (1_{\bar{a}^l} \otimes f \otimes 1_{\bar{a}^m}) \circ (1_{\bar{a}^l} \otimes c_{\bar{a}^m, b})$$



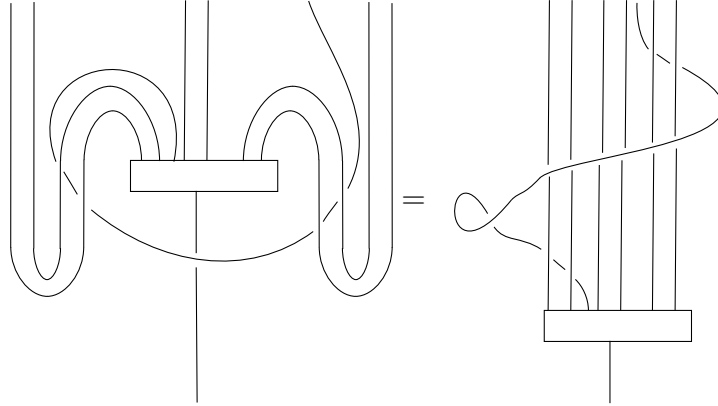
Let $G : \mathcal{C} \boxtimes \tilde{\mathcal{C}} \rightarrow Z(\mathcal{C})$ be the usual inclusion. Then

$$G(\bar{a}^{l+m} \boxtimes \tilde{b}) = (\bar{a}^{l+m} \otimes b, (c_{\bar{a}^{m+l}, \cdot} \otimes 1_b) \circ (1_{\bar{a}^l} \otimes \tilde{c}_{b, \cdot})).$$

If we identify $\mathcal{C} \boxtimes \tilde{\mathcal{C}}$ with its image under G , then we have

$$(V_{l,m}^n)^{-1} \rho_{n-(m+l), a}^{\bar{a}^{l+m} \boxtimes \tilde{b}} V_{l,m}^n = \theta_a \pi_a^b(A_{l,m}^n)$$

The picture is given by



$\theta_a^{-1} \pi_a^b(A_{l,m}^n)$ is diagonalizable with basis $\{(V_{l,m}^n)^{-1}(b_i)\}$. This implies

Corollary 4.1. $\pi_a(A_{l,m}^n)$ is diagonalizable and $\pi_a(A_{l,m}^n)^{N(n-(m+l))} = 1$. Let ω be a $(n - (m + l))N^{th}$ root of unity. Then $\theta_a^{-1}\omega$ occurs as an eigenvalue of $\pi_{n,a}^b(A_{l,m}^n)$ with multiplicity $K_{n-(m+1), \omega}^{\bar{a}^{l+m} \boxtimes \tilde{b}}$

Thus the computation of the eigenvalues $A_{l,m}^n$ reduces to computing the eigenvalues of appropriate rotation operators, which we know how to do in terms of Ng-Schauenburg indicators. If \mathcal{C} happens to be modular, we can compute these eigenvalues in terms of the modular data S and T , as we demonstrate explicitly in the next section.

Now we suppose \mathcal{C} is modular with modular data S, T , with $\text{ord}(T) = M$. In this case, $G : \mathcal{C} \boxtimes \tilde{\mathcal{C}} \rightarrow Z(\mathcal{C})$ is an equivalence. Again, assume a is fixed so $K_{n, \omega}^b$ refers to the multiplicity

of ω in $\rho_{n,a}^b$, where ω is a some nM^{th} root of unity, and $b \in Z(\mathcal{C})$. Specializing the formulas from above to $n = 2$, we see that for ω a $2M^{th}$ root of unity we have

$$K_{2,\omega}^{c\boxtimes \tilde{b}} = \frac{\delta(\omega^2, \frac{\theta_b}{\theta_c})}{2} \left(\omega^{-1} \nu_{2,1}^{c\boxtimes \tilde{b}}(a) + N_{\bar{c},a,a}^b \right).$$

Recall from section 2 that $S_{c\boxtimes b, d\boxtimes e} = S_{c,d} S_{\bar{b},e}$, and $\theta_{a\boxtimes \tilde{b}} = \frac{\theta_a}{\theta_b}$, so we have the following formula:

$$\nu_{2,1}^{c\boxtimes \tilde{b}}(a) = \sum_{d,e \in \text{Irr}(\mathcal{C})} \left(\frac{\theta_d}{\theta_e} \right)^2 S_{c,d} S_{\bar{b},e} N_{d,e}^a$$

Combining these formulas gives

Proposition 4.2. *Let \mathcal{C} be a modular category with modular data S, T , and let ω be a $2M^{th}$ root of unity. Then $\rho_{2,a}^{a\boxtimes \tilde{b}}$ contains eigenvalue ω with multiplicity*

$$K_{2,\omega}^{c\boxtimes \tilde{b}} := \delta_{\omega^2 = \frac{\theta_b}{\theta_c}} \frac{1}{2} \left(\omega^{-1} \sum_{d,e \in \text{Irr}(\mathcal{C})} \left(\frac{\theta_d}{\theta_e} \right)^2 S_{c,d} S_{\bar{b},e} N_{d,e}^a + N_{\bar{c},a,a}^b \right)$$

In particular, $K_{2,\omega}^{c\boxtimes \tilde{b}}$ is a non-negative integer, and $\omega \in \text{Spec}(\rho_{2,a}^{c\boxtimes \tilde{b}})$ if and only if $K_{2,\omega}^{c\boxtimes \tilde{b}} > 0$.

It appears somewhat surprising that this number is a non-negative integer in general. Using the Verlinde formula, we can compute the terms $N_{\bar{c},a,a}^b$ in the sum in terms of the S -matrix, and thus the above formula is entirely expressible in terms of the modular data. In B_∞ , σ_i is conjugate to $A_{0,0}^2$ and $\sigma_i \sigma_{i+1} \sigma_i$ is conjugate to $A_{1,0}^3$. This implies

- (1) $\text{Spec}(\pi_a(\sigma_i)) = \{\theta_a^{-1} \omega : K_{2,\omega}^{1\boxtimes \tilde{b}} \neq 0 \text{ for some } b \in \mathcal{C}\}.$
- (2) $\text{Spec}(\pi_a(\sigma_i \sigma_{i+1} \sigma_i)) = \{\theta_a^{-1} \omega : K_{2,\omega}^{\bar{a} \boxtimes \tilde{b}} \neq 0 \text{ for some } b \in \mathcal{C}\}.$

Remark 4.3. Note that this allows us to compute the link invariants for the closures of powers of braids $A_{m,l}^n$ in terms of generalized FS-indicators, hence also in terms of the modular data when \mathcal{C} is modular. For example, for the closure of the braid $(A_{0,0}^p)^q$ is the (p, q) torus link. And thus above observations given a formula for this invariant in terms of the dimensions, twists, and $\nu_{p,q}^b$. This class of examples is directly computable from the GFS-indicators (without first determining the eigenvalues), but other examples will require more interesting interactions with the eigenvalues themselves. It would be interesting to determine general classes of links whose invariants can be obtained this way, and precise formulas in terms of the modular data.

4.1. An example: $Z(H)$. As an example, we consider the quadratic fusion category H , described in the previous section (this is the standard categorification corresponding to the even part of the Haagerup subfactor). Then the modular data for the center of this category is well known, and was first computed by Izumi in [Izu01]. We use here the form from [EG11]:

$$S := \frac{1}{3} \begin{pmatrix} x & 1-x & 1 & 1 & 1 & 1 & y & y & y & y & y & y \\ 1-x & x & 1 & 1 & 1 & 1 & -y & -y & -y & -y & -y & -y \\ 1 & 1 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & -y & 0 & 0 & 0 & 0 & c(1) & c(2) & c(3) & c(4) & c(5) & c(6) \\ y & -y & 0 & 0 & 0 & 0 & c(2) & c(4) & c(6) & c(5) & c(3) & c(1) \\ y & -y & 0 & 0 & 0 & 0 & c(3) & c(6) & c(4) & c(1) & c(1) & c(5) \\ y & -y & 0 & 0 & 0 & 0 & c(4) & c(5) & c(1) & c(3) & c(6) & c(2) \\ y & -y & 0 & 0 & 0 & 0 & c(5) & c(3) & c(2) & c(6) & c(1) & c(4) \\ y & -y & 0 & 0 & 0 & 0 & c(6) & c(1) & c(5) & c(2) & c(4) & c(3) \end{pmatrix}$$

$$T := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{2i\pi}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\frac{2i\pi}{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{12i\pi}{13}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{4i\pi}{13}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{4i\pi}{13}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{10i\pi}{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{12i\pi}{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{10i\pi}{13}} \end{pmatrix}$$

where $x = \frac{3-\sqrt{13}}{26}$, $y = \frac{3}{\sqrt{13}}$, and $c(j) = -2y \cos \frac{2\pi j}{13}$

As a demonstration of our formula, we consider the the 6th simple object, x_i , where the objects are indexed left to right along the columns of the T -matrix. We chose this object because it has multiplicity 2 objects in its tensor decomposition, but our formula is easy to compute for any objects. Then $\theta_6 = e^{-\frac{2i\pi}{3}}$, and the two entries in the “possible eigenvalues column” below are $\omega_i \theta_6^{-1}$, where $\omega_i = \pm \sqrt{\theta_i}$, and these are the possible eigenvalues of the braid acting on $H(x_i, x_6 \otimes x_6)$. The final column indicates the multiplicity as computed by our formula.

object	possible eigenvalues	multiplicities
1	$(-e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}})$	(0, 1)
2	$(-e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}})$	(1, 1)
3	$(-e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}})$	(0, 1)
4	$(-e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}})$	(0, 1)
5	(1, -1)	(1, 0)
6	$(-e^{\frac{\pi i}{3}}, e^{\frac{\pi i}{3}})$	(2, 0)
7	$(e^{\frac{5\pi i}{39}}, -e^{\frac{5\pi i}{39}})$	(1, 0)
8	$(-e^{\frac{20\pi i}{39}}, -e^{\frac{20\pi i}{39}})$	(1, 0)
9	$(-e^{\frac{32\pi i}{39}}, e^{\frac{32\pi i}{39}})$	(1, 0)
10	$(e^{\frac{2\pi i}{39}}, -e^{\frac{2\pi i}{39}})$	(0, 1)
11	$(-e^{\frac{8\pi i}{39}}, e^{\frac{8\pi i}{39}})$	(1, 0)
12	$(-e^{\frac{11\pi i}{39}}, e^{\frac{11\pi i}{39}})$	(0, 1)

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